Chapter 12
Rolling, Torque, and Angular Momentum

If you forget everything in this chapter, be proud that you at least remember:

The angular momentum of any closed system is always constant. It will never change no matter what you do. This is the big and powerful equation of -> $L_i = L_f$. Do not forget this!

Key concepts:

Kinetic Energy of a rolling object-

Okay, this chapter really isn't too bad. Take kinetic energy of a rolling object for example. If the object is rotating, then the Kinetic energy equality is $K = \frac{1}{2} I w^2$ where *I* is the rotational inertia of the object rotating around the axis through its center of mass and *w* is the angular speed of the object. And if you have a pure translational moving object (no rotation) then Kinetic energy found by $K = \frac{1}{2} M v^2$. Notice the remarkable similarities. The nice thing about uniform objects is that the center of mass is smack dab in the center.

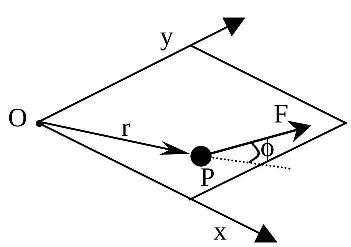
So if you have a rotating translational object, also known as a rolling object, then you just add the two equations together to get total Kinetic energy.

- $K = \frac{1}{2} I w^2 + \frac{1}{2} M v^2$

Remember that *I* is the rotational inertia of the object through the center of mass and *v* is the velocity of the object through the center of mass.

Torque-

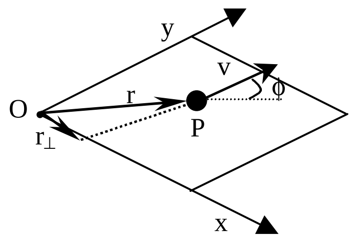
Torque is defined as $t = r \times F$. This applies to any particle that moves in any path in relation to a fixed point, not just rotating systems.



P is a particle in this closed system. It position is at a distance r from the reference point of *O*. *O* can be any reference point and the distance to the particle from that point will be *r*. The torque created by the force *F* on particle *P* (in relation to point *O*) is given by $r \times F$ or $rF \sin f$ also known as $rF_{\perp}$. Remember that this just isn't only for rotating objects. Every object has a chance to share in this equation.

- $t = r \times F = rF \sin f$

Angular Momentum-



Angular momentum is defined as $\ell = r \times p = m(r \times v)$.

Given a particle *P* with mass *m*, its angular momentum (in reference to point *O*) is found by the equation $\ell = m(r \times v)$ or $\ell = mrv\sin f = mvr_{\perp}$. You can think of $r_{\wedge}$ as the closest particle *P* will pass to point *O* with the given velocity. Choosing a good reference point, such as point *O*, will become important when dealing with Equilibrium type of equations as seen from Chapter 13.

- $\ell = m(r \times v) = mrv\sin f$

Newton's Second Law for Angular Momentum-

Just like Newton's Second law when dealing with forces, remember $\sum F = \frac{dp}{dt}$, Angular momentum has its own version of this law, which is $\sum t = \frac{d\ell}{dt}$. This is for a single particle. However for a system of particles the equation is pretty much the same thing $\sum t_{ext} = \frac{dL}{dt}$ where $L = \sum_{i=1}^{n} \ell_i$. For a system of particles, the total angular momentum is equal to the total of all the angular momentum of each particle in the system.

- $\sum t = \frac{d\ell}{dt}$
- $\sum t_{ext} = \frac{dL}{dt}$

Angular Momentum of a Rigid Body rotating on a fixed axis-

If an object is just spinning around its axis, then its angular momentum is defined by $L = Iw$. Notice the similarities to the linear momentum equation $P = Mv$. Freaky, isn't it?

- $L = Iw$

Major Problem Types-

Rolling down ramps-

For any object with a rotational inertia of *I*, the acceleration of the object down an incline with an angle f is given by the equation $a = \dfrac{g \sin f}{1 + I / MR^2}$.

Finding Angular Momentum of an object with respect to a point-

Any moving object or particle has an angular momentum with respect to a point as defined by the equation $\ell = mrv \sin f = mvr_{\perp}$ where *m* is the mass of the object, *v* is the velocity of it, *r* is the distance between the object and the reference point, and f is the angle of the velocity to the vector *r*.

A change in angular velocity of an object in a closed system-

Problems like this are usually characterized by an object rotating on a fixed axis with a certain angular velocity and its rotational inertia changes to give it a new angular velocity. Two ways an object can change its rotational inertia are by redistributing its mass, like growing, shrinking, or by changing its configuration or by changing is mass altogether, such as having an object added to it, or mass taken away.

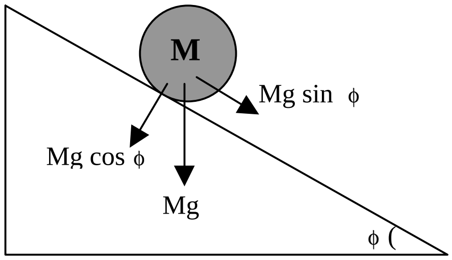
Because angular momentum is always constant you can use the equation $I_o w = I_f w$ where I_o is the object's starting rotational inertia and I_f is its final rotational inertia.

Inelastic collisions that spin an object about its axis-

These problems are easy to recognize because they involve a moving object hitting a stationary object that will end up rotating because of the collision. The moving object will usually end up sticking to the rotating one (it makes things way easier).

The moving object (sometimes a bullet or a rock) has an angular momentum with respect to any point. The best point to pick is the point of rotation of the soon to be spinning object. Since the moving object is the only thing moving, it has all the angular momentum so $L_i = m(r \times v) = mvr \sin f$. When it hits the stationary object, it will start spinning and has an angular momentum of Iw . I of course depends on several things which are usually supplied, include the mass of the object, *M*, and the mass of the first object, *m*. This will make the total mass equal to *M*+*m* (this is important, don't forget it). So, now the final angular momentum of the system is $L_f = Iw_f$. Because angular momentum never changes in a system, then $L_i = L_f$ which means $mvr \sin f = Iw_f$. The only tricky thing to put in here is to give the rotating object an intitial angular velocity of ω_o which means $L_i = mvr \sin f + I_o w_o$ where I_o is the object's rotational inertia before the small object is added to it, so its mass is just *M*. This leads to a final equation of $mvr \sin f + I_o w_o = I_f w_f$.

A uniform sphere rolls down an incline. (a) What must be the incline angle if the linear acceleration of the center of the sphere is to be 0.10g? (b) For this angle, what would be the acceleration of a frictionless block sliding down the incline?



Solution:

(a). This is a rolling down a ramp problem, so the equation of choice is $a = \dfrac{g\sin f}{1 + I/MR^2}$ and you want $a = 0.1g$.

For this problem use the equation $I = \frac{2}{5}MR^2$. The sphere has a mass M and radius R. so the final equation is

$0.1g = \dfrac{g\sin f}{1 + I/MR^2}$ Starting equation

$0.1g = \dfrac{g\sin f}{1 + \dfrac{2MR^2}{5MR^2}}$ Expand the *I*

$0.1 = \dfrac{\sin f}{1 + \dfrac{2}{5}}$ *g* cancels from both sides and the MR_2 cancel

$$\sin f = 0.1\left(1 + \frac{2}{5}\right)$$

$$f = \sin^{-1}\left(0.1\left(\frac{7}{5}\right)\right)$$

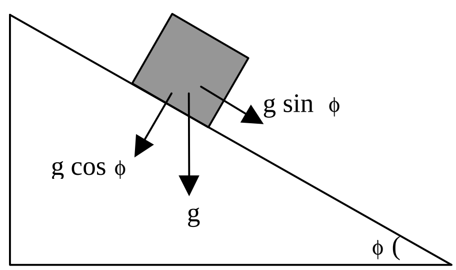
$$f = 8.0^\circ$$

(b) This is just an easy kinematics problem

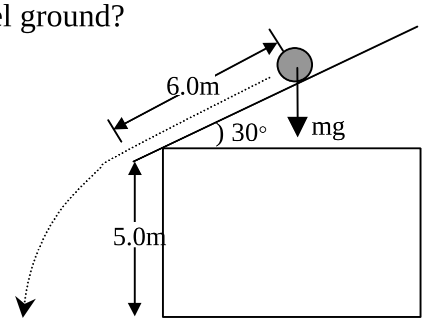
$a = g\sin f$

$a = g\sin(8^\circ)$

$\boxed{a = 0.14g = 1.37 m/s^2}$



A solid cylinder of radius 10cm and mass 12 kg starts from rest and rolls without slipping a distance of 6m down a house roof that is inclined at 30°. (a) What is the angular speed of the cylinder about its center as it leaves the house roof? (b) The outside wall of the house is 5m high. How far from the edge of the roof does the cylinder hit the level ground?



Solution:

(a) This is rolling down a ramp equation where $a = \dfrac{g\sin q}{1 + I/MR^2}$

$v = \sqrt{2ad}$	-Velocity of the objet after accelerating for a distance *d*.
$\omega_o = v/R = \sqrt{2ad}\ / R$	-This is the angular speed equation
$I = ½ MR^2$	-Rotational inertia for a cylinder
$a = g \sin\theta/(1+ I_{cm}/(MR^2))$	-Equation 12-9
$a = g \sin\theta / (1 + ½)$	-MR^2 cancels with the one from the *I*
$a = 2/3\ g \sin\theta$	-The acceleration of the object

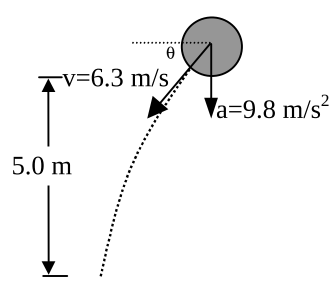
$\omega_o = (\sqrt{2ad})/R = \dfrac{\sqrt{((4/3)gd\sin q)}}{R}$ Substitute for *a* into equation

$\omega_o = \dfrac{\sqrt{(4/3)(9.8)(6)(\sin 30°)}}{0.1}$ Solve for ω

$\boxed{\omega_o = 63 \text{ rad/s}}$

(b) This is just a projectile motion equation

$h = v_y t + ½at^2$
$v_y = -v \sin\theta$
$h = -(v \sin\theta)t - ½gt^2$
$-5 = -(6.3 \sin 30)t - 4.9t^2$ (Use quadratic to solve)
$t = 0.7386$ s



$x = v_x t$
$v_x = v \cos\theta$
$x = (v \cos\theta)t$
$x = (6.3 \cos 30)(0.7386)$
$\boxed{x = 4.0 \text{ m}}$

What is the net torque about the origin on a flea located at coordinates (0, -4m, 5m) when forces F_1 = (3N)**k** and F_2 = (-2N)**j** acting on the flea?

Solution:

This is just a torque about a point problem and you should use $t = r \times F$. The flea is at (0, -4, 5) which is the moment arm, *r*. The force vector is the 3**k** + -2**j** which is equal to (-2, 3). To cross the two vectors use the formula:

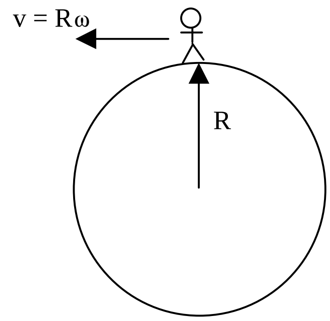
$(a_yb_z-a_zb_y)$**i** +$(a_zb_x-a_xb_z)$**j** +$(a_xb_y-a_yb_x)$**k** where *a* is the *r* vector and *b* is the *F* vector. The **j** and **k** values go to zero because the *x* component of both vectors are zero.

$\tau = r \text{ x } F_{net} = [-4\mathbf{j} + 5\mathbf{k}] \text{ x } [3\mathbf{k} + -2\mathbf{j}]$

$\tau = (-4(3)-(5(-2)))\mathbf{i}$

$\boxed{\tau = (-2\mathbf{i}) \text{ N}\bullet\text{m}}$

What is the magnitude of the angular momentum, about the Earth's center, of an 84kg person on the equator due to the rotation of the Earth? (The radius of the Earth is $6.37 \text{ x } 10^6$ meters.)



Solution:

The Earth makes a rotation every day which is 24 hours long.

$24hrs \times \frac{60\min}{1hr} \times \frac{60\sec}{1\min} = 86400\sec$. This is the time it takes for 1 revolution.

$$w = \frac{2p}{T}$$

$\ell = mRv\sin 90^\circ$ The angular momentum equation

$$v = Rw$$

$$= mR^2w$$

$$= (84)(6.37\times 10^6)^2\left(\frac{2p}{86400}\right)$$

$$\boxed{= 2.5 \text{ x } 10^{11} \text{ kg}\bullet\text{m}^2/\text{s}}$$

A 2 kg object moves in a plane with velocity components $v_x = 30$m/s, $v_y = 60$m/s, and it passes through the point (x,y) = (3, -4)m. (a) What is its angular momentum relative to the origin at this moment? (b) What is its angulare momentum relative to the point (-2, -2)m at this same moment?

Solution:

This is just a finding angular momentum of a particle problem and you should use $\ell = m(r \times v)$. The object is at (3**i**, -4**j**) which is the moment arm, *r*. The velocity vector is (30**i** + 60**j**). To cross the two vectors use the formula: $(a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$ where *a* is the *r* vector and *b* is the *v* vector. The **i** and **j** values go to zero because the *z* component of both vectors are zero.

$$\ell = m(r \times v)$$
$$= m(r_x v_y - r_y v_x)$$
$$= (2)(3(60) + 4(30))$$
$$\boxed{= (6 \text{ x } 10^2 \text{ kg} \bullet \text{m}^2/\text{s})\mathbf{k}}$$

A projectile of mass m is fired from the ground with an initial speed v_0 and an initial angle θ above the horizontal. (a) Find an expression for the magnitude of its angular momentum about the firing point as a function of time. (b) Find the rate at which the angular momentum changes with time. (c) Evaluate the magnitude of r x F directly and compare the result with (b). Why should the results be identical?

Solution:

(a) This is a kinematics problem with some angular momentum stuff thrown in to make it a little harder. The moment arm, *r*, is going to change with the position of the projectile. Its vector is given as $(v_o \cos\theta)\mathbf{i} + (v_o \sin\theta)\mathbf{j}$. Use the equation $d_x = voxt$ for the distance in the *x* direction and $d_y = v_{oy}t - \frac{1}{2}gt^2$ for the *y* direction. Set *r* equal to this.

$r = (v_{0x}t)\mathbf{i} + (v_{0y}t - ½gt^2)\mathbf{j} = (v_0 \cos\theta\, t)\mathbf{i} + (v_0 \sin\theta - ½gt^2)\mathbf{j}$

Next the velocity is constant for the x direction but for the y direction the velocity is $v = v_{oy} - gt = v_0 \sin\theta - gt$. Set the velocity vector equal to this for both x and y directions.

$v = v_x\,\mathbf{i} + v_y\,\mathbf{j} = (v_0 \cos\theta)\mathbf{i} + (v_0 \sin\theta - gt)\mathbf{j}$

Cross the two together using the formula $\ell = m(r \times v)$ and then simplify.

$\boxed{l = m(\mathbf{r} \times \mathbf{v}) = -½mv_0 \cos\theta\, gt^2\mathbf{k}}$

(b) Just take the derivative of the above equation knowing that $-½mv_0 \cos\theta\, g$ is a constant and that $\frac{d(t^2)}{dt} = 2t$. Multiply the two together and you get …

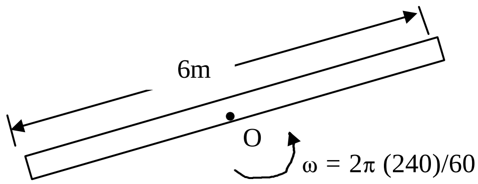
$\boxed{dl/dt = -v_0\, mgt \cos\theta_0\, \mathbf{k}}$

(c) The only force acting on the projectile is its weight, *mg*, which acts downward. Cross this with the **r** vector obtained earlier to get the torque about the launching point.

$\tau = r \times F = [(v_0 \cos\theta_0\, t)\mathbf{i} + r_y\,\mathbf{j}] \times (-mg\,\mathbf{j}) = \boxed{-v_0\, mgt \cos\theta_0\, \mathbf{k}}$

(this is the same as in part b and this is because $dl/dt = \tau = r \times F$)

A uniform rod rotates in a horizontal plane about a vertical axis through one end. The rod is 6m long, weighs 10N, and rotates at 240 rev/min clockwise when seen from above. Calculate (a) the rotational inertia of the rod about the axis of rotation and (b) the angular momentum of the rod about that axis.

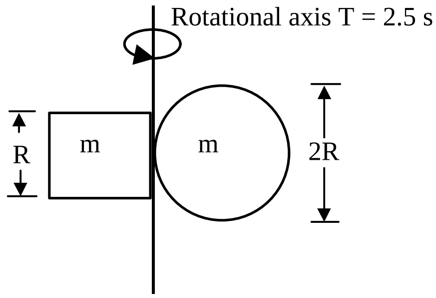


<u>Solution:</u>

(a) The weight of the rod also gives the mass of the rod because W = mg.
So the mass is W / g. The *I* of the rod equal to 1/3 ML^2.
$I = 1/3\ Ml^2 = (1/3)[(10N)/(9.8m/s^2)](6^2)$
$\boxed{I = 12.2\ kg \bullet m^2}$

(b) Angular momentum is defined as *Iw*. Remember that *w* is 2π * (rev/sec).
$L = I\omega = (12.2\ kg \bullet m^2)[(2\pi)(240)/(60s)]$
$\boxed{L = 308\ kg \bullet m^2/s \text{ downward}}$

The figure shows a rigid structure consisting of a circular hoop, of radius R and mass m, and a square made of four thin bars, each of length R and mass m. The rigid structure rotates at a constant speed about a vertical axis with a period of rotation of 2.5s. Assuming R = .5m and m = 2 kg, calculate (a) the structures rotational inertia about the axis of rotation and (b) its angular momentum about that axis.



<u>Solution:</u>

(a) The moment of inertia for the Square will the two thin rods perpendicular to the axis of rotation and the thin bar that is not on the axis of rotation (The other bar is entirely at a distance of zero from the axis and has a rotational inertia of zero).

Square (moment of inertia): bar on axis of rotation does not contribute

Other vertical bar = mR^2

Each horizontal bar = $1/3\ mR^2$

Total = $mR^2 + 2/3\ mR^2$

For the hoop, use the parallel axis theorem once the rotational inertia has been found.

Hoop (moment of inertia) = $mR^2 + ½mR^2 = 3/2\ mR^2$

$I = 3/2\ mR^2 + mR^2 + (2/3)mR^2 = (19/6)mR^2 = (19/6)(2kg)(.5m)^2$

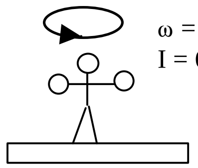
$\boxed{I = 1.6\ kg \bullet m^2}$

(b) Now, with the rotational inertia found, the angular momentum is easy. Just remember to convert the period into angular speed with the equation $\frac{2p}{T} w$.

$L = I\omega = I(2\pi/T) = [2\pi(1.6\ kg \bullet m^2\ )]\ /\ 2.5s$

$\boxed{L = 4\ kg \bullet m^2}$

A man stands on a platform that is rotating (without friction) with an angular speed of 1.2 rev/s; his arms are outstretched and he holds a weight in each hand. The rotational inertia of the system of man, weights, and platform about the central axis is 6 $kg \bullet m^2$. If by moving the weights the man decreases the rotational inertia of the system to 2.0 $kg \bullet m^2$, (a) what is the resulting angular speed of the platform and (b) what is the ratio of the new kinetic energy of the system to the original kinetic energy? (c) What provided the added kinetic energy?



Solution:

(a)

The total angular momentum of the system of man, weights, & platform is conserved

Li = Lf

$I_i w_i = I_f w_f$

$\omega_f = (I_i / I_f)\omega_i = [(6\ kg \bullet m^2) / (2\ kg \bullet m^2)](1.2\ rev/s)$

$\boxed{\omega_f = 3.6\ rev/s}$

(b) Remember that the Kinetic energy of a rotating body is $\frac{1}{2} I w$.

$K_i = \frac{1}{2} I_i \omega_i^2$

$K_f = \frac{1}{2} I_f \omega_f^2$

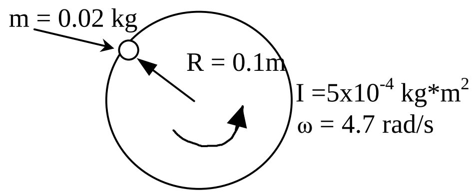
$K_f / K_i = (I_f \omega_f^2) / (I_i \omega_i^2)$

$K_f / K_i = [(2 kg \bullet m^2)(3.6\ rev/s)^2] / [(6\ kg \bullet m^2)(1.2\ rev/s)^2]$

$\boxed{K_f / K_i = 3.0}$

(c) The added kinetic energy came from the man doing work in decreasing the rotational inertia by moving the weights closer to his body.

A phonograph record of mass 0.1kg and radius 0.1m rotates about a vertical axis through its center with an angular speed of 4.7 rad/s. The rotational inertia of the record about its axis of rotation is 5×10^{-4} kg•m^2 . A wad of putty of mass 0.02 kg drops vertically onto the record from above and sticks to the edge of the record. What is the angular speed of the record immediately after the putty sticks to it?



<u>Solution:</u>

Angular momentum is always conserved, so with the increase in mass to the record, the angular speed should now be slower. The putty is a point mass with a rotational inertia of Mr^2.

Putty: mass m

Record: mass M, and radius r

$L_i = I_i \omega_i = L_f = I_f \omega_f$ Initial momentum must be equal to the final momentum.

$\omega_f = (I_i \omega_i)/I_f = (I_i \omega_i) / (I_i + mr^2)$ Solve for ω_f.

$\omega_f = [5 \times 10^{-4}$ kg•m^2)(4.7 rad/s)] / [5x10^{-4} kg•m^2 + (.02kg)(.1m)^2

$\boxed{\omega_f = 3.4 \text{ rad/s}}$